

ON POLYMORPHISM-HOMOGENEOUS RELATIONAL STRUCTURES AND THEIR CLONES

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ABSTRACT. A relational structure is homomorphism-homogeneous if every homomorphism between finite substructures extends to an endomorphism of the structure. This notion was introduced recently by Cameron and Nešetřil. In this paper we consider a strengthening of homomorphism-homogeneity — we call a relational structure polymorphism-homogeneous if every partial polymorphism with a finite domain extends to a global polymorphism of the structure. It turns out that this notion (under various names and in completely different contexts) has been existing in algebraic literature for at least 30 years. Motivated by this observation, we dedicate this paper to the topic of polymorphism-homogeneous structures. We study polymorphism-homogeneity from a model-theoretic, an algebraic, and a combinatorial point of view. E.g., we study structures that have quantifier elimination for positive primitive formulae, and show that this notion is equivalent to polymorphism-homogeneity for weakly oligomorphic structures. We demonstrate how the Baker-Pixley theorem can be used to show that polymorphism-homogeneity is a decidable property for finite relational structures. Eventually, we completely characterize the countable polymorphism-homogeneous graphs, the polymorphism-homogeneous posets of arbitrary size, and the countable polymorphism-homogeneous strict posets.

1. INTRODUCTION

A relational structure is called polymorphism-homogeneous if every partial polymorphism with finite domain extends to a global polymorphism of the structure. The phenomenon of polymorphism-homogeneity appears in different contexts under varying names in the algebraic literature. The earliest occurrence of this idea seems to be the Baker-Pixley Theorem in universal algebra [1] that simultaneously generalizes the Chinese remainder theorem and Langrange's interpolation theorem. In [23], motivated by questions from multivalued logics and clone theory, Romov studies relational structures over finite sets for which every partial polymorphism can be extended to a global one. In [24] he extended this approach to countably infinite structures. Another source of the notion of polymorphism-homogeneity is [13], where Kaarli characterizes all polymorphism-homogeneous meet-complete lattices of equivalence relations and, using this characterization, identifies classes of locally affine complete algebras. Related to polymorphism-homogeneity is also the interpolation condition (IC) that plays an important role in the

2010 *Mathematics Subject Classification.* 08A02 (03C10, 03C40, 05C63, 06A06).

Key words and phrases. relational structure, polymorphism-clone, relational clone, weak oligomorphy, quantifier elimination, homomorphism-homogeneity, polymorphism-homogeneity, Galois connection.

theory of natural dualities [4]. A structure has the (IC) if every partial polymorphism (not necessary with finite domain) extends to a global one. Thus, structures that have the (IC) are in particular polymorphism-homogeneous.

The appearance of one and the same idea in such diverse contexts motivated us to have a closer look onto the theory of polymorphism-homogeneous structures and its connections with other, related, model-theoretic notions like quantifier elimination. Our results extend and generalize previous results by Romov [24]. A diagram that sums up our findings can be found at the end of Section 3. In Section 4 we apply the results from Section 3, to characterize all those countable relational structures that have the property that the primitively positively definable relations coincide with the relations that are invariant under all polymorphisms of the given structure. As a consequence, we can give a short, model-theoretic proof of Romov's characterization of locally closed relational clones on countable sets [24, Thm.3.5]. In Section 5 some connections between the notion of polymorphism-homogeneity and the Baker-Pixley Theorem are presented. In particular, we will derive an algebraic characterization of the structures that fulfill the interpolation condition among all polymorphism-homogeneous structures. Moreover, we show that for finite structures polymorphism-homogeneity is decidable.

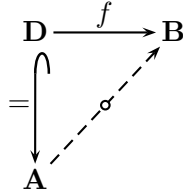
Recently, in their seminal paper [3], Cameron and Nešetřil studied various generalizations of the classical notion of (ultra-)homogeneity (recall that a relational structure is called homogeneous if every isomorphism between finite substructures extends to an automorphism of the given structure). One of the discussed generalizations is homomorphism-homogeneity. A relational structure is homomorphism-homogeneous if and only if every homomorphism between finite substructures extends to an endomorphism of the structure. It soon turned out that this notion is very relevant in the theory of transformation monoids on countable sets [5, 6, 19, 20, 21]. Moreover, there exists already a rich classification theory for homomorphism-homogeneous structures [2, 7, 11, 12, 16, 17, 18]. Note that by definition, every polymorphism-homogeneous structure is also homomorphism-homogeneous. Thus, in a sense, the polymorphism-homogeneous structures are especially beautiful homomorphism-homogeneous structures. In Section 6, we will completely characterize the countable polymorphism-homogeneous graphs, the countable polymorphism-homogeneous strict posets, and all polymorphism-homogeneous non-strict posets. At the end of the paper we shortly review Kaarli's elegant characterization of polymorphism-homogeneous meet-complete lattices of equivalence relations.

2. PRELIMINARIES

Relational Structures. A relational signature L is a family $(\varrho_i)_{i \in I}$ of relational symbols, together with a function $\text{ar} : L \rightarrow \mathbb{N} \setminus \{0\}$ that assigns to every symbol ϱ its arity $\text{ar}(\varrho)$. If ϱ is a relational symbol that belongs to L , then we write $\varrho \in L$. A relational structure \mathbf{A} over the signature L is a pair $(A, (\varrho_{\mathbf{A}})_{\varrho \in L})$ such that A is a set and $\varrho_{\mathbf{A}}$ is a relation of arity $\text{ar}(\varrho)$ on A . The set A is also called the *carrier* of \mathbf{A} . Relational structures over L will simply be called L -structures, or structures (if L is clear from the

context). Homomorphisms, isomorphisms, automorphisms, epimorphisms, and monomorphisms are defined as usual, for L -structures. Embeddings are strong monomorphisms (in the model theoretic sense, cf. [10]). Also, we use the term *substructure* in the model-theoretic sense. In the graph-theoretic terminology, model-theoretic substructures are usually called induced substructures. Recall that the age of a relational structure \mathbf{A} is the class of all finite structures that embed into \mathbf{A} . In other words, a structure is in the age of \mathbf{A} if and only if it is isomorphic to some finite substructure of \mathbf{A} . The age of \mathbf{A} will be denoted by $\text{Age}(\mathbf{A})$.

Polymorphism-homogeneity. Let \mathbf{A} and \mathbf{B} be relational structures over the same signature L . Let $\mathbf{D} \leq \mathbf{A}$. Then a homomorphism $f : \mathbf{D} \rightarrow \mathbf{B}$ is called *partial homomorphism* from \mathbf{A} to \mathbf{B} with domain \mathbf{D} (written as $f : \mathbf{A} \dashrightarrow \mathbf{B}$).



The structure \mathbf{D} will usually be denoted by $\text{dom } f$. In the special case where L is the empty signature, partial homomorphisms are just usual partial functions. Partial homomorphisms of a structure to itself are called *partial endomorphisms*. Finally, partial homomorphisms with a finite domain will be called *local homomorphisms*.

Let I be a set and, for $i \in I$, let $\mathbf{A}_i = (A_i, (\varrho_{\mathbf{A}_i})_{\varrho \in L})$ be relational structures. The *product* of the family $(\mathbf{A}_i)_{i \in I}$ is defined by

$$\mathbf{A} = \prod_{i \in I} \mathbf{A}_i := \left(\prod_{i \in I} A_i, (\varrho_{\mathbf{A}})_{\varrho \in L} \right),$$

where

$$\prod_{i \in I} A_i := \{(a_i)_{i \in I} \mid \forall i \in I : a_i \in A_i\}$$

and such that for all $\varrho \in R$ we have

$$\varrho_{\mathbf{A}} := \{((a_{1,i})_{i \in I}, \dots, (a_{\text{ar}(\varrho),i})_{i \in I}) \mid \forall i \in I : (a_{1,i}, \dots, a_{\text{ar}(\varrho),i}) \in \varrho_{\mathbf{A}_i}\}.$$

If $\emptyset \subsetneq J \subseteq I$, then we denote the *projection homomorphism* with respect to J by

$$e_J : \prod_{i \in I} \mathbf{A}_i \rightarrow \prod_{i \in J} \mathbf{A}_i \quad (a_i)_{i \in I} \mapsto (a_i)_{i \in J}.$$

In the special case, when $J = \{j\}$, we write e_j instead of $e_{\{j\}}$.

When all \mathbf{A}_i are equal to one and the same structure \mathbf{A} , then we abbreviate the product $\prod_{i \in I} \mathbf{A}_i$ by \mathbf{A}^I . This special kind of direct product is called a *direct power* of \mathbf{A} . Direct powers will usually occur for the special case $I = k$, where k is a finite cardinal number.

Let L be a relational signature and let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in L})$ be an L -structure. Then the k -ary *polymorphisms* of \mathbf{A} are defined to be the homomorphisms from \mathbf{A}^k to \mathbf{A} . Partial and local k -ary polymorphisms of \mathbf{A} are defined accordingly, as partial or local homomorphisms from \mathbf{A}^k to \mathbf{A} ,

respectively. The set of all polymorphisms of \mathbf{A} will be denoted by $\text{Pol}(\mathbf{A})$, while the set of all k -ary polymorphisms will be denoted by $\text{Pol}^{(k)}(\mathbf{A})$.

It is not hard to see that k -ary partial polymorphisms of \mathbf{A} are characterized by the following property:

Lemma 2.1. *A partial function $f : A^k \multimap A$ is a partial polymorphism of \mathbf{A} if and only if for all $\varrho \in R$ and for all $\bar{a}_1, \dots, \bar{a}_{\text{ar}(\varrho)} \in \text{dom } f$ with $\bar{a}_i = (a_{i,1}, \dots, a_{i,l})$ holds that*

$$\begin{bmatrix} a_{1,1} \\ \vdots \\ a_{\text{ar}(\varrho),1} \end{bmatrix} \in \varrho \mathbf{A}, \dots, \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{\text{ar}(\varrho),k} \end{bmatrix} \in \varrho \mathbf{A} \implies \begin{bmatrix} f(a_{1,1}, \dots, a_{1,k}) \\ \vdots \\ f(a_{\text{ar}(\varrho),1}, \dots, a_{\text{ar}(\varrho),k}) \end{bmatrix} \in \varrho \mathbf{A}$$

□

We say that a relational structure \mathbf{A} is *k -polymorphism-homogeneous* if every k -ary local polymorphism of \mathbf{A} can be extended to a polymorphism of \mathbf{A} . If \mathbf{A} is k -polymorphism-homogeneous for every $k \in \mathbb{N} \setminus \{0\}$, then we say that \mathbf{A} is *polymorphism-homogeneous*. If a structure \mathbf{A} is 1-polymorphism-homogeneous then we call it *homomorphism-homogeneous*. Thus, every polymorphism-homogeneous structure is homomorphism-homogeneous. However, polymorphism-homogeneity is a much stronger property than homomorphism-homogeneity, as can be seen from the following simple observation:

Proposition 2.2. *A structure \mathbf{A} is k -polymorphism-homogeneous if and only if \mathbf{A}^k is homomorphism-homogeneous.* □

An immediate consequence is that a relational structure \mathbf{A} is polymorphism-homogeneous if and only if all finite powers of \mathbf{A} are homomorphism-homogeneous.

The following proposition will show that the concept of k -polymorphism-homogeneity defines a decreasing hierarchy on the homomorphism-homogeneous structures with homomorphism-homogeneous structures on the top.

Proposition 2.3. *Let \mathbf{A} be a relational structure, and let $k \in \mathbb{N} \setminus \{0\}$. If \mathbf{A} is $(k+1)$ -polymorphism-homogeneous, then it is also k -polymorphism-homogeneous.*

Proof. Let f be a k -ary local polymorphism of \mathbf{A} with domain \mathbf{D} . Moreover, let \mathbf{D}_k be the (finite) substructure of \mathbf{A} that is induced by $e_{k-1}(D)$. First we construct a $(k+1)$ -ary local polymorphism $\hat{f} : \mathbf{A}^{k+1} \multimap \mathbf{A}$ by setting

$$\hat{f}(a_0, \dots, a_{k-1}, x) := f(a_0, \dots, a_{k-1}) \quad \text{for all } x \in D_k.$$

More precisely, \hat{f} is the unique homomorphism that makes the following diagram commutative:

$$(1) \quad \begin{array}{ccccc} \mathbf{A}^k & \xleftarrow{=} & \mathbf{D} & \xrightarrow{f} & \mathbf{A} \\ \uparrow e_{\{0, \dots, k-1\}} & & \uparrow e & \nearrow \hat{f} & \\ \mathbf{A}^{k+1} & \xleftarrow{=} & \mathbf{D} \times \mathbf{D}_k & & \end{array}$$

where $e : \mathbf{D} \times \mathbf{D}_k \rightarrow \mathbf{D}$ is the projection homomorphism onto \mathbf{D} . So in particular $\hat{f} = f \circ e$, and $\text{dom } \hat{f} = \mathbf{D} \times \mathbf{D}_k$ is finite. Since \mathbf{A} is $(k+1)$ -polymorphism-homogeneous, \hat{f} can be extended to a polymorphism \hat{g} of \mathbf{A} . Now we can define a polymorphism $g : \mathbf{A}^k \rightarrow \mathbf{A}$: For this consider the embedding

$$\iota : \mathbf{A}^k \hookrightarrow \mathbf{A}^{k+1} \quad (a_0, \dots, a_{k-1}) \mapsto (a_0, \dots, a_{k-1}, a_{k-1}).$$

and its restriction $\iota|_{\mathbf{D}} : \mathbf{D} \rightarrow \mathbf{D} \times \mathbf{D}_k$. Note that ι and $\iota|_{\mathbf{D}}$ are right-inverses of $e_{\{0, \dots, k-1\}}$ and e , respectively. Hence the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{A}^k & \xleftarrow{=} & \mathbf{D} & \xrightarrow{f} & \mathbf{A} \\ \downarrow \iota & & \downarrow \iota|_{\mathbf{D}} & \nearrow \hat{f} & \\ \mathbf{A}^{k+1} & \xleftarrow{=} & \mathbf{D} \times \mathbf{D}_k & & \\ & \searrow \hat{g} & & & \end{array}$$

Now, with $g = \hat{g} \circ \iota$, and using that the above diagram commutes, we obtain that g is indeed an extension of f to a polymorphism of \mathbf{A} . \square

Let \mathbf{A} be a relational structure. Then \mathbf{A} is called *weakly polymorphism-homogeneous* if for all $n \in \mathbb{N} \setminus \{0\}$, for every finite structure $\mathbf{C} \leq \mathbf{A}^n$, for all $\mathbf{B} \leq \mathbf{C}$ and for all homomorphisms $h : \mathbf{B} \rightarrow \mathbf{A}$ there exists a homomorphism $\hat{h} : \mathbf{C} \rightarrow \mathbf{A}$ that makes the following diagram commutative:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{h} & \mathbf{A} \\ \downarrow \cong & \nearrow \hat{h} & \\ \mathbf{C} & & \end{array}$$

Clearly, polymorphism-homogeneity implies weak polymorphism-homogeneity. Moreover, for countable relational structures the concepts of polymorphism-homogeneity and weak polymorphism-homogeneity coincide.

3. THE GALOIS CONNECTIONS BETWEEN RELATIONS AND PRIMITIVE POSITIVE TYPES

Positive primitive types. Let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in L})$ be a relational structure over the relational signature L . Following [10], with $L_{\omega\omega}$ we denote the full first order language with predicate symbols from L .

As usual, every formula $\varphi(x_1, \dots, x_m)$ from $L_{\omega\omega}$ defines an m -ary relation $\varphi^{\mathbf{A}}$ on A where $\bar{a} \in \varphi^{\mathbf{A}}$ if φ holds in \mathbf{A} when interpreting x_i through a_i , for $i = 1, \dots, m$. In that case, we say that \bar{a} fulfills φ in \mathbf{A} write $\mathbf{A} \models \varphi(\bar{a})$.

A set of positive primitive formulae in $L_{\omega\omega}$ with free variables in $\{x_1, \dots, x_m\}$ is called an *m -ary positive primitive type*. For a relation $\varrho \subseteq A^m$ we define $\text{Tp}_{\mathbf{A}}(\varrho)$ to be the set of all primitive positive formulae $\varphi(x_1, \dots, x_m)$ from $L_{\omega\omega}$ such that $\varrho \subseteq \varphi^{\mathbf{A}}$. On the other hand, for an m -ary positive primitive type Ψ , we define

$$\Psi^{\mathbf{A}} := \bigcap_{\varphi \in \Psi} \varphi^{\mathbf{A}}.$$

The operators $\varrho \mapsto \text{Tp}_\mathbf{A}(\varrho)$ and $\Psi \mapsto \Psi^\mathbf{A}$ form a Galois-connection between the m -ary relations on A and the m -ary positive primitive types over $L_{\omega\omega}$. The Galois-closed types are called *closed positive primitive types* of \mathbf{A} .

A closed m -ary positive primitive type over \mathbf{A} is called *principal* if it is generated by one of its elements. Types that are of the shape $\text{Tp}_\mathbf{A}(\tau)$ for some finite relation $\tau \subseteq A^m$ are called *complete m -ary positive primitive types* over \mathbf{A} .

If Ψ is an m -ary positive primitive type, then by $\Psi^{(k)}$ we denote the set of all formulae from Ψ that are logically equivalent to a primitive positive formula of the shape

$$\exists x_{m+1}, \dots, x_{m+k} \psi(x_1, \dots, x_{m+k}),$$

where $\psi(x_1, \dots, x_{m+k})$ is a conjunction of atoms. Instead of $(\text{Tp}_\mathbf{A}(\tau))^{(k)}$ we will write also $\text{Tp}_\mathbf{A}^{(k)}(\tau)$.

If Ψ is a principal positive primitive type over \mathbf{A} , then the relation $\Psi^\mathbf{A}$ is called *positive primitively definable* over \mathbf{A} . The set of all positive primitively definable relations (of all arities) over \mathbf{A} is called the *relational algebra* generated by \mathbf{A} . It is denoted by $[\mathbf{A}]_{\text{RA}}$, while its m -ary part is denoted by $[\mathbf{A}]_{\text{RA}}^{(m)}$.

In general, the set of Galois-closed relations (of arbitrary arities) over \mathbf{A} will be denoted by $\overline{[\mathbf{A}]_{\text{RA}}}$.

Algebraic closure systems. Recall that a closure system \mathcal{C} on a set A is called *algebraic* if there exists some algebraic structure \mathbf{A} with carrier A whose closure system of universes of subalgebras is equal to \mathcal{C} .

A closure system is called *inductive* if it is closed with respect to unions of chains of closed sets. There is another equivalent way to define inductive closure systems. A subset M of a given closure system is called *upwards directed* if any two elements of M are contained in a third. Now it is easy to see that a closure system is inductive if and only if it is closed with respect to unions of upwards directed sets of closed sets.

We have the following classical result that characterizes algebraic closure systems (cf. [26]):

Theorem 3.1 (Schmidt [26, Hauptsatz]). *Let \mathcal{C} be a closure system on a set A , and let $C : \mathcal{P}(A) \rightarrow \mathcal{C}$ be the associated closure operator. Then the following are equivalent:*

- (1) \mathcal{C} is an algebraic closure system,
- (2) \mathcal{C} is an inductive closure system,
- (3) $\forall S \subseteq A : C(S) = \bigcup_{\substack{T \subseteq S \\ T \text{ finite}}} C(T)$. □

Weak oligomorphy and related properties. Let \mathbf{A} be a relational structure. We say that

- \mathbf{A} is *weakly oligomorphic* if for all $m \in \mathbb{N} \setminus \{0\}$ there are just finitely many m -ary positive primitively definable relations over \mathbf{A} ,
- \mathbf{A} is *algebraic* if for all $m \in \mathbb{N}$ the closure-system $[\mathbf{A}]_{\text{RA}}^{(m)}$ is algebraic,
- \mathbf{A} is *pp-atomic* if all complete positive primitive types over \mathbf{A} are principal,

- \mathbf{A} has *principal positive primitive types*, if all closed positive primitive types over \mathbf{A} are principal.

Lemma 3.2. *All weakly oligomorphic relational structures are algebraic, pp-atomic, and have principal positive primitive types. Moreover, if a relational structure has principal positive primitive types, then it is also pp-atomic. \square*

Lemma 3.3. *Let \mathbf{A} be a relational structure. Then the following are equivalent.*

- (1) \mathbf{A} has *principal positive primitive types*.
- (2) $[\mathbf{A}]_{\text{RA}} = \overline{[\mathbf{A}]_{\text{RA}}}$.
- (3) $[\mathbf{A}]_{\text{RA}}^{(m)}$ is closed with respect to arbitrary intersections for every $m \in \mathbb{N} \setminus \{0\}$.

Proof. (1) \Rightarrow (2) Suppose that \mathbf{A} has principal positive primitive types. Note that $[\mathbf{A}]_{\text{RA}}^{(m)} \subseteq \overline{[\mathbf{A}]_{\text{RA}}^{(m)}}$ always holds. Let $\sigma \in \overline{[\mathbf{A}]_{\text{RA}}^{(m)}}$, and let $\Psi = \text{Tpp}_{\mathbf{A}}(\sigma)$. Since Ψ is closed, it follows that there exists a $\psi \in \Psi$ such that $\text{Tpp}_{\mathbf{A}}(\psi^{\mathbf{A}}) = \Psi$. However, from this it follows that $\psi^{\mathbf{A}} = \Psi^{\mathbf{A}} = \sigma$. Hence $\sigma \in [\mathbf{A}]_{\text{RA}}^{(m)}$.

(2) \Rightarrow (3) Take any family $\{\sigma_i\}_{i \in I}$ of elements of $[\mathbf{A}]_{\text{RA}}^{(m)}$. We know that

$$\sigma := \bigcap_{i \in I} \sigma_i \in \overline{[\mathbf{A}]_{\text{RA}}^{(m)}}.$$

So from the premise it follows that $\sigma \in [\mathbf{A}]_{\text{RA}}^{(m)}$.

(3) \Rightarrow (1) Let Ψ be an arbitrary closed primitive positive type over \mathbf{A} , and let $\sigma = \Psi^{\mathbf{A}}$. Then

$$\sigma = \bigcap_{\psi \in \Psi} \psi^{\mathbf{A}}.$$

Since $[\mathbf{A}]_{\text{RA}}^{(m)}$ is closed with respect to arbitrary intersections, we get that $\sigma \in [\mathbf{A}]_{\text{RA}}^{(m)}$. Hence, there exists a formula $\psi \in \Psi$ such that $\sigma = \psi^{\mathbf{A}}$. Thus, $\Psi^{\mathbf{A}} = \sigma = \psi^{\mathbf{A}}$. This shows that Ψ is principal. \square

Polylocality. A relational structure \mathbf{A} will be called *k-polylocal* if for every $m \in \mathbb{N}$ and for every $\sigma \subseteq A^m$ holds

$$\sigma \in \overline{[\mathbf{A}]_{\text{RA}}^{(m)}} \iff \forall \tau \subseteq A^m \text{ finite, } \bar{b} \in A^m : \\ (\tau \subseteq \sigma \text{ and } \text{Tpp}_{\mathbf{A}}^{(k)}(\bar{b}) \subseteq \text{Tpp}_{\mathbf{A}}^{(k)}(\tau)) \Rightarrow \bar{b} \in \sigma.$$

If \mathbf{A} is 0-polylocal, then we call it *polylocal*.

Lemma 3.4. *A relational structure \mathbf{A} is k-polylocal if and only if for every $\sigma \subseteq A^m$ holds*

$$\sigma \in \overline{[\mathbf{A}]_{\text{RA}}^{(m)}} \iff \sigma = \bigcup_{\substack{\tau \subseteq \sigma \\ \tau \text{ finite}}} (\text{Tpp}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}}$$

\square

Lemma 3.5. *Let \mathbf{A} be a relational structure. Then the following are equivalent:*

- (1) \mathbf{A} is k -polylocal,
 (2) \mathbf{A} is algebraic and for all $m \in \mathbb{N} \setminus \{0\}$, and for all finite $\tau \subseteq A^m$ we have

$$(\text{TpP}_{\mathbf{A}}(\tau))^{\mathbf{A}} = (\text{TpP}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}}$$

Proof. (1) \Rightarrow (2) : Let $m \in \mathbb{N} \setminus \{0\}$, $\tau \subseteq A^m$ finite. By the definition of k -polylocality, we have

$$(\text{TpP}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}} \subseteq (\text{TpP}_{\mathbf{A}}(\tau))^{\mathbf{A}}.$$

On the other hand, $\text{TpP}_{\mathbf{A}}^{(k)}(\tau) \subseteq \text{TpP}_{\mathbf{A}}(\tau)$ implies that $(\text{TpP}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}} \supseteq (\text{TpP}_{\mathbf{A}}(\tau))^{\mathbf{A}}$. Thus, (2) is proved.

(2) \Rightarrow (1) Since \mathbf{A} is algebraic, we have.

$$\sigma \in \overline{[\mathbf{A}]_{\text{RA}}^{(m)}} \iff \sigma = \bigcup_{\substack{\tau \subseteq \sigma \\ \tau \text{ finite}}} (\text{TpP}_{\mathbf{A}}(\tau))^{\mathbf{A}}$$

By assumption we may replace in each term of the union $\text{TpP}_{\mathbf{A}}(\tau)$ by $\text{TpP}_{\mathbf{A}}^{(k)}(\tau)$. This together with Lemma 3.4 proves the claim. \square

Proposition 3.6. *Let \mathbf{A} be an algebraic, weakly polymorphism-homogeneous relational structure. Then \mathbf{A} is polylocal.*

Proof. We are going to use Lemma 3.5 in order to show that \mathbf{A} is polylocal. Let $\tau \subseteq A^m$ be finite. In particular, $\tau = \{\bar{a}_1, \dots, \bar{a}_l\}$, for some $l \in \mathbb{N}$, with $\bar{a}_i = (a_{i,1}, \dots, a_{i,m})$.

Let $\bar{b} = (b_1, \dots, b_m) \in (\text{TpP}_{\mathbf{A}}^{(0)}(\tau))^{\mathbf{A}}$, i.e., $\text{TpP}_{\mathbf{A}}^{(0)}(\tau) \subseteq \text{TpP}_{\mathbf{A}}^{(0)}(\bar{b})$. Let us show that then also $\text{TpP}_{\mathbf{A}}(\tau) \subseteq \text{TpP}_{\mathbf{A}}(\bar{b})$. For some $k \in \mathbb{N}$ let $\psi \in \text{TpP}_{\mathbf{A}}^{(k)}(\tau)$. It is well-known that there exists a finite structure \mathbf{D} and a tuple $\bar{d} \in D^m$ such that for all $\bar{a} \in A^m$ we have that $\bar{a} \in \psi^{\mathbf{A}}$ if and only if there is a homomorphism $f : (\mathbf{D}, \bar{d}) \rightarrow (\mathbf{A}, \bar{a})$ — i.e., $f : \mathbf{D} \rightarrow \mathbf{A}$ and $f(d_i) = a_i$, for $i \in \{1, \dots, m\}$. Since $\tau \subseteq \psi^{\mathbf{A}}$, it follows that there exists a homomorphism $g : (\mathbf{D}, \bar{d}) \rightarrow (\mathbf{A}^l, \bar{c})$, where $\bar{c} = (\bar{c}_1, \dots, \bar{c}_m)$ and where $\bar{c}_j = (a_{1,j}, \dots, a_{l,j})$ for $1 \leq j \leq m$.

Let now $\widetilde{\mathbf{M}} = g(\mathbf{D})$, $\tilde{g} : \mathbf{D} \rightarrow \widetilde{\mathbf{M}}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{g} & \mathbf{A}^l \\ \downarrow \tilde{g} & \nearrow \text{=} & \\ \widetilde{\mathbf{M}} & & \end{array}$$

Let $M = \{\bar{c}_1, \dots, \bar{c}_m\}$, and $B = \{b_1, \dots, b_m\}$. Let \mathbf{M} be the substructure of \mathbf{A}^m induced by M , and let \mathbf{B} be the substructure of \mathbf{A} induced by B . Since $\text{TpP}_{\mathbf{A}}^{(0)}(\tau) \subseteq \text{TpP}_{\mathbf{A}}^{(0)}(\bar{b})$, the function $\tilde{f} : \mathbf{M} \rightarrow \mathbf{B}$ defined by $\tilde{f} : \bar{c}_i \mapsto b_i$ for $1 \leq i \leq m$, is an epimorphism. Let f be the homomorphism that makes the

following diagram commutative:

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{f} & \mathbf{A} \\ \downarrow \tilde{f} & \searrow \scriptstyle = & \uparrow \\ \mathbf{B} & & \end{array}$$

Since \mathbf{A} is weakly polymorphism-homogeneous, it follows that f has an extension to $\widetilde{\mathbf{M}}$ — call it h . Let $\widetilde{\mathbf{B}} = h(\widetilde{\mathbf{M}})$, and let \tilde{h} be the homomorphism that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{\mathbf{M}} & \xrightarrow{h} & \mathbf{A} \\ \downarrow \tilde{h} & \searrow \scriptstyle = & \uparrow \\ \mathbf{B} & & \end{array}$$

Then, by construction, the following diagram commutes:

$$\begin{array}{ccc} (\widetilde{\mathbf{M}}, \bar{c}) & \xrightarrow{h} & (\mathbf{A}, \bar{b}) \\ \downarrow \tilde{h} & \searrow \scriptstyle = & \uparrow \\ (\mathbf{B}, \bar{b}) & & \end{array}$$

Summing up our findings, we obtain that also the following diagram commutes:

$$\begin{array}{ccc} (\mathbf{D}, \bar{d}) & \xrightarrow{g} & (\mathbf{A}', \bar{c}) \\ \downarrow \tilde{g} & \searrow \scriptstyle = & \uparrow \\ (\widetilde{\mathbf{M}}, \bar{c}) & & \\ \downarrow \tilde{h} & \searrow h & \\ (\mathbf{B}, \bar{b}) & \xrightarrow{\scriptstyle =} & (\mathbf{A}, \bar{b}) \end{array}$$

In particular, $h \circ \tilde{g} : (\mathbf{D}, \bar{d}) \rightarrow (\mathbf{A}, \bar{b})$. Hence, $\bar{b} \in \psi^{\mathbf{A}}$. Consequently, $\psi \in \text{Tp}_{\mathbf{A}}(\bar{b})$, whence also $\text{Tp}_{\mathbf{A}}(\tau) \subseteq \text{Tp}_{\mathbf{A}}(\bar{b})$, i.e. $\bar{b} \in (\text{Tp}_{\mathbf{A}}(\tau))^{\mathbf{A}}$. It follows that $(\text{Tp}_{\mathbf{A}}^{(0)}(\tau))^{\mathbf{A}} \subseteq (\text{Tp}_{\mathbf{A}}(\tau))^{\mathbf{A}}$. Moreover, from $\text{Tp}_{\mathbf{A}}^{(0)}(\tau) \subseteq \text{Tp}_{\mathbf{A}}(\tau)$ it follows that $(\text{Tp}_{\mathbf{A}}(\tau))^{\mathbf{A}} \subseteq (\text{Tp}_{\mathbf{A}}^{(0)}(\tau))^{\mathbf{A}}$. Now from Lemma 3.5 it follows that \mathbf{A} is polylocal. \square

Proposition 3.7. *Let \mathbf{A} be a pp-atomic relational structure that is polylocal or has the property that every complete positive primitive type Ψ over \mathbf{A} is generated by $\Psi^{(0)}$. Then \mathbf{A} is weakly polymorphism-homogeneous.*

Proof. Let $\mathbf{B} \leq \mathbf{A}^k$ be finite, and let $f : \mathbf{B} \rightarrow \mathbf{A}$. Let us enumerate the elements of B like $B = \{\bar{b}_1, \dots, \bar{b}_m\}$. Assume $\bar{b}_i = (b_{i,1}, \dots, b_{i,k})$, for $1 \leq i \leq m$. Define $\tau := \{(b_{1,j}, \dots, b_{m,j}) \mid 1 \leq j \leq k\}$, and $\bar{c} := (c_1, c_2, \dots, c_m)$, where

$c_i = f(\bar{b}_i)$. Then $\text{Tpp}_{\mathbf{A}}^{(0)}(\tau) \subseteq \text{Tpp}_{\mathbf{A}}^{(0)}(\bar{c})$. By our assumptions, it follows that $\text{Tpp}_{\mathbf{A}}(\tau) \subseteq \text{Tpp}_{\mathbf{A}}(\bar{c})$.

Suppose that $\widehat{\mathbf{B}}$ is a finite superstructure of \mathbf{B} in \mathbf{A} . Then we can enumerate the elements of \widehat{B} like $\widehat{B} = \{\bar{b}_1, \dots, \bar{b}_m, \bar{b}_{m+1}, \dots, \bar{b}_{m+n}\}$. Let $\sigma := \{(b_{1,j}, \dots, b_{m+n,j}) \mid 1 \leq j \leq k\}$, and let $\Psi := \text{Tpp}_{\mathbf{A}}(\sigma)$. Since \mathbf{A} is pp-atomic, there exists a positive primitive formula φ such that $\text{Tpp}_{\mathbf{A}}(\varphi^{\mathbf{A}}) = \Psi$. Clearly, we have that $(\exists x_{m+1} \dots \exists x_{m+n} \varphi) \in \text{Tpp}_{\mathbf{A}}(\tau)$. Thus also $(\exists x_{m+1} \dots \exists x_{m+n} \varphi) \in \text{Tpp}_{\mathbf{A}}(\bar{c})$. That means that there exist c_{m+1}, \dots, c_{m+n} in A , such that

$$\varphi \in \text{Tpp}_{\mathbf{A}}(c_1, \dots, c_m, c_{m+1}, \dots, c_{m+n}).$$

But from this it follows that $\text{Tpp}_{\mathbf{A}}((c_1, \dots, c_{m+n})) \supseteq \Psi = \text{Tpp}_{\mathbf{A}}(\sigma)$. Consequently, the mapping $\bar{b}_i \mapsto c_i$ ($1 \leq i \leq m+n$) defines a homomorphism that extends f to $\widehat{\mathbf{B}}$.

This shows that \mathbf{A} is weakly polymorphism-homogeneous. \square

Proposition 3.8. *Let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in R})$ be an algebraic relational structure. If every complete positive primitive type Ψ over \mathbf{A} is generated by $\Psi^{(k)}$, then \mathbf{A} is k -polylocal.*

Proof. Let $\tau \subseteq A^m$ be finite. Then $\text{Tpp}_{\mathbf{A}}(\tau)$ is a complete positive primitive type over \mathbf{A} . Hence, by the assumption we have that $\text{Tpp}_{\mathbf{A}}(\tau)$ is generated by $\text{Tpp}_{\mathbf{A}}^{(k)}(\tau)$. In particular, we conclude that $(\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}} = (\text{Tpp}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}}$. Hence, by Lemma 3.5 we have that \mathbf{A} is k -polylocal. \square

Positive primitive elimination sets. Let \mathbf{A} be an L -structure, and let Φ be a set of positive primitive formulae from $L_{\omega\omega}$. We call Φ a *primitive positive elimination set* for \mathbf{A} if for every positive primitive formula $\varphi(x_1, \dots, x_m) \in L_{\omega\omega}$ there exists a finite set $\Psi \subseteq \Phi$ such that all formulae from Ψ have their free variables in the set $\{x_1, \dots, x_m\}$ and such that $\varphi^{\mathbf{A}} = \Psi^{\mathbf{A}}$.

Specifically, we say that \mathbf{A} has *quantifier elimination for positive primitive formulae* if the set of quantifier free positive primitive formulae from $L_{\omega\omega}$ is a positive primitive elimination set for \mathbf{A} .

Proposition 3.9. *Let \mathbf{A} be a weakly oligomorphic k -polylocal relational structure. Then the set of all primitive positive formulae of quantifier depth at most k from $L_{\omega\omega}$ is a positive primitive elimination set for \mathbf{A} .*

Proof. Let $m \in \mathbb{N} \setminus \{0\}$, and let $\varphi(x_1, \dots, x_m) \in L_{\omega\omega}$ be primitive positive. Take $\varrho := \varphi^{\mathbf{A}}$. Since \mathbf{A} is weakly oligomorphic, there exists a finite subset τ of ϱ such that $(\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}} = \varrho$. By Lemma 3.5 we have $(\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}} = (\text{Tpp}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}}$. Again using that \mathbf{A} is weakly oligomorphic, there exists a $\psi \in \text{Tpp}_{\mathbf{A}}^{(k)}(\tau)$ such that $\psi^{\mathbf{A}} = (\text{Tpp}_{\mathbf{A}}^{(k)}(\tau))^{\mathbf{A}} = \varrho = \varphi^{\mathbf{A}}$. \square

Proposition 3.10. *Let \mathbf{A} be a relational structure that has a positive primitive elimination set of formulae of quantifier depth at most k . Then every closed positive primitive type Ψ of \mathbf{A} is generated by $\Psi^{(k)}$.*

Proof. Let Ψ be an m -ary closed positive primitive type over \mathbf{A} . For every $\psi \in \Psi$, let Ψ_{ψ} be a finite set of positive primitive formulae of quantifier depth

$\leq k$ with free variables in $\{x_1, \dots, x_m\}$ such that $\psi^{\mathbf{A}} = (\Psi_\psi)^{\mathbf{A}}$. Define $\Psi^* := \bigcup_{\psi \in \Psi} \Psi_\psi$. Then, by construction, $\text{Tpp}_{\mathbf{A}}(\Psi^{\mathbf{A}}) = \text{Tpp}_{\mathbf{A}}((\Psi^*)^{\mathbf{A}})$. Moreover, since $\Psi^* \subseteq \Psi^{(k)}$, we have $\text{Tpp}_{\mathbf{A}}((\Psi^*)^{\mathbf{A}}) \subseteq \text{Tpp}_{\mathbf{A}}((\Psi^{(k)})^{\mathbf{A}}) \subseteq \text{Tpp}_{\mathbf{A}}(\Psi^{\mathbf{A}}) = \Psi$. Hence $\text{Tpp}_{\mathbf{A}}((\Psi^{(k)})^{\mathbf{A}}) = \Psi$. \square

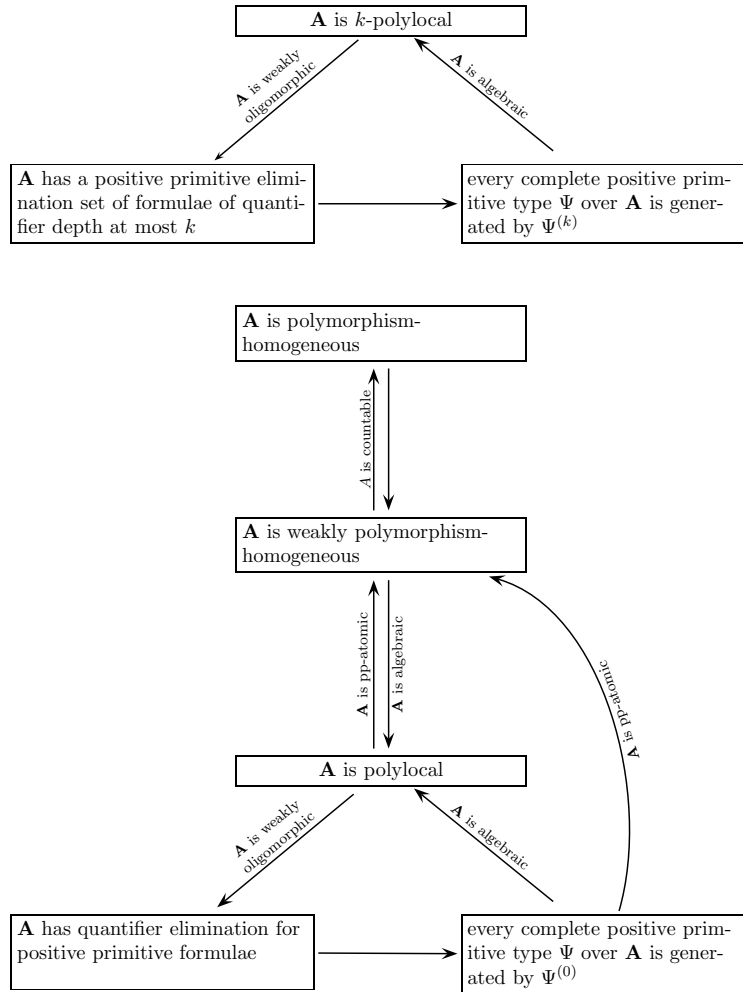
Corollary 3.11. *A weakly oligomorphic relational structure has quantifier elimination for positive primitive formulae if and only if it is weakly polymorphism-homogeneous.*

Proof. Let \mathbf{A} be a weakly oligomorphic relational structure. Then, in particular, \mathbf{A} is algebraic and pp-atomic.

“ \Rightarrow ” By Proposition 3.10, every complete positive primitive type Ψ over \mathbf{A} is generated by $\Psi^{(0)}$. From Proposition 3.7 it follows that \mathbf{A} is weakly polymorphism-homogeneous.

“ \Leftarrow ” From Proposition 3.6 it follows that \mathbf{A} is polylocal. Now, by Proposition 3.9 we have that \mathbf{A} has quantifier elimination for positive primitive formulae. \square

Collected findings. The results of this section can be visualized by two diagrams:



4. CHARACTERIZING POSITIVE PRIMITIVELY DEFINABLE RELATIONS

In the following, with $R_A^{(m)}$ we will denote the set of all m -ary relations on A . Moreover, we define

$$R_A := \bigcup_{m \in \mathbb{N} \setminus \{0\}} R_A^{(m)}.$$

A relational algebra W on a set A is a subset of R_A with the property that

$$W = [\mathbf{A}]_{\text{RA}}$$

for some relational structure \mathbf{A} on A .

Given any set W of relations on A , we can construct a *canonical relational structure* \mathbf{C}_W : We take as relational signature L the set W itself. The arity of each element of L is the arity of this element, considered as a relation. Moreover, for every $\varrho \in L$ we define $\varrho_{\mathbf{C}_W} := \varrho$. It is easy to see that W is a relational algebra if and only if $W = [\mathbf{C}_W]_{\text{RA}}$. Moreover, in general $[\mathbf{C}_W]_{\text{RA}}$ is the smallest relational algebra that contains W . We will write $[W]_{\text{RA}}$ instead of $[\mathbf{C}_W]_{\text{RA}}$. Moreover, instead of $\text{Pol}(\mathbf{C}_W)$ we will write $\text{Pol}(W)$.

As usual, we define $O_A^{(n)}$ to be the set of all function from A^n to A , and we set

$$O_A := \bigcup_{n \in \mathbb{N} \setminus \{0\}} O_A^{(n)}.$$

If $f \in O_A^{(n)}$, and $\varrho \subseteq A^m$ we call ϱ *invariant* for f (and we write $f \triangleright \varrho$ whenever f is a polymorphism of $\mathbf{C}_{\{\varrho\}}$). In other words, $f \triangleright \varrho$ if and only if for all $\bar{a}_1, \dots, \bar{a}_m \in A^m$ with $\bar{a}_i = (a_{i,1}, \dots, a_{i,n})$ holds that

$$\begin{bmatrix} a_{1,1} \\ \vdots \\ a_{m,1} \end{bmatrix} \in \varrho, \dots, \begin{bmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{bmatrix} \in \varrho \implies \begin{bmatrix} f(a_{1,1}, \dots, a_{1,n}) \\ \vdots \\ f(a_{m,1}, \dots, a_{m,n}) \end{bmatrix} \in \varrho.$$

For any subset $F \subseteq O_A$ we define

$$\text{Inv}(F) := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\},$$

and for any $m \in \mathbb{N} \setminus \{0\}$, we define $\text{Inv}^{(m)}(F) := \text{Inv}(F) \cap R_A^{(m)}$. The operators Inv and Pol form a Galois-connection between sets of functions and sets of relations on A . It is well-known that every Galois-closed set of relations forms a relational algebra. In general the Galois-closed sets of relations are called *relational clones*. It is one of the fundamental theorems of general algebra that whenever A is a finite set then the relational clones on A and the relational algebras on A coincide. However, when A is infinite, then there exist relational algebras that are not relational clones. A relational algebra W is a relational clone if and only if $W = \text{Inv}(\text{Pol}(W))$.

A *clone* on A is a subset of O_A that contains all projections and that is closed with respect to composition. Every Galois closed set of functions is a clone. In general, Galois-closed sets of functions on A are called *local clones*. Another fundamental theorem of general algebra states that on a finite set A all clones are local. However, when A is infinite, then there are clones that are not local. Standard references for clones on finite sets are [22, 27, 14]. For a survey of known results about clones on infinite sets we refer to [9].

If for some relational structure \mathbf{A} we have $[\mathbf{A}]_{\text{RA}} = \text{Inv}(\text{Pol}(\mathbf{A}))$, then this means that a relation ϱ on A is positive primitively definable over \mathbf{A} if and only if it is invariant for all polymorphisms of \mathbf{A} .

Theorem 4.1. *Let \mathbf{A} be a countable algebraic polymorphism-homogeneous relational structure, that has principal positive primitive types. Then $[\mathbf{A}]_{\text{RA}} = \text{Inv}(\text{Pol}(\mathbf{A}))$.*

Proof. We will show that for every $m \in \mathbb{N}$

$$\text{Inv}^{(m)}(\text{Pol}(\mathbf{A})) = [\mathbf{A}]_{\text{RA}}^{(m)}.$$

Since \mathbf{A} is an algebraic, polymorphism-homogeneous relational structure, from Proposition 3.6 it follows that \mathbf{A} is polylocal. Since \mathbf{A} has principal positive primitive types, by Lemma 3.3 we have that $[\mathbf{A}]_{\text{RA}}^{(m)} = \overline{[\mathbf{A}]_{\text{RA}}^{(m)}}$. Now it is enough to show that

$$(2) \quad \forall \tau \subseteq A^m \text{ finite} : (\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}} = \Gamma_{\text{Pol}(\mathbf{A})}(\tau),$$

where $\Gamma_{\text{Pol}(\mathbf{A})}(\tau)$ denotes the smallest invariant relation of $\text{Pol}(\mathbf{A})$ that contains τ . Indeed, if this equality holds, then for every relation $\sigma \in \text{Inv}^{(m)}(\text{Pol}(\mathbf{A}))$ we have that

$$\sigma = \bigcup_{\substack{\tau \subseteq \sigma \\ \tau \text{ finite}}} \Gamma_{\text{Pol}(\mathbf{A})}(\tau) = \bigcup_{\substack{\tau \subseteq \sigma \\ \tau \text{ finite}}} (\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}} \in \overline{[\mathbf{A}]_{\text{RA}}^{(m)}} = [\mathbf{A}]_{\text{RA}}^{(m)},$$

since \mathbf{A} is algebraic.

Let us now proceed by proving (2). First of all, since $(\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}} \in [\mathbf{A}]_{\text{RA}}^{(m)} \subseteq (\text{Inv}(\text{Pol}(\mathbf{A})))^{(m)}$, it follows that $\Gamma_{\text{Pol}(\mathbf{A})}(\tau) \subseteq (\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}}$. On the other hand, for $\bar{b} \in (\text{Tpp}_{\mathbf{A}}(\tau))^{\mathbf{A}}$ we have that in particular, $\text{Tpp}_{\mathbf{A}}^{(0)}(\bar{b}) \supseteq \text{Tpp}_{\mathbf{A}}^{(0)}(\tau)$. Suppose now that $\tau = \{\bar{a}_1, \dots, \bar{a}_n\}$, with $\bar{a}_i = (a_{i,1}, \dots, a_{i,m})$, and let \mathbf{B} be the substructure of \mathbf{A}^n induced by $\{(a_{1,j}, \dots, a_{n,j}) \mid 1 \leq j \leq m\}$. We define the mapping $f : \mathbf{B} \rightarrow \mathbf{A}$ by $f(a_{1,j}, \dots, a_{n,j}) := b_j$, for $j \in \{1, \dots, m\}$. It is easy to see that this defines a local polymorphism of \mathbf{A} . Since \mathbf{A} is polymorphism-homogeneous, it follows that f can be extended to an polymorphism \hat{f} of \mathbf{A} . But, from this it follows at once that $\bar{b} \in \Gamma_{\text{Pol}(\mathbf{A})}(\tau)$. \square

Theorem 4.2. *Let \mathbf{A} be a countable relational structure. Then $[\mathbf{A}]_{\text{RA}} = \text{Inv}(\text{Pol}(\mathbf{A}))$ if and only if \mathbf{A} is algebraic and has principal positive primitive types.*

Proof. (\Leftarrow) Let $\hat{\mathbf{A}}$ be the canonical structure of $[\mathbf{A}]_{\text{RA}}$. Clearly, then $[\hat{\mathbf{A}}]_{\text{RA}} = [\mathbf{A}]_{\text{RA}}$, and $\hat{\mathbf{A}}$ has quantifier elimination for positive primitive formulae. Hence, by Proposition 3.10, all closed positive primitive types over $\hat{\mathbf{A}}$ are generated by their quantifier free part. Clearly, $\hat{\mathbf{A}}$ is also algebraic and has principal positive primitive types.

From Proposition 3.7 it follows that $\hat{\mathbf{A}}$ is weakly polymorphism-homogeneous. As A is countable, it follows that $\hat{\mathbf{A}}$ is polymorphism-homogeneous. Finally, from Theorem 4.1 we obtain that

$$[\mathbf{A}]_{\text{RA}} = [\hat{\mathbf{A}}]_{\text{RA}} = \text{Inv}(\text{Pol}(\hat{\mathbf{A}})) = \text{Inv}(\text{Pol}(\mathbf{A})).$$

(\Rightarrow) In general, we have

$$[\mathbf{A}]_{\text{RA}} \subseteq \overline{[\mathbf{A}]_{\text{RA}}} \subseteq \text{Inv}(\text{Pol}(\mathbf{A})).$$

So, if $[\mathbf{A}]_{\text{RA}} = \text{Inv}(\text{Pol}(\mathbf{A}))$, then we have in particular, that $[\mathbf{A}]_{\text{RA}} = \overline{[\mathbf{A}]_{\text{RA}}}$. Hence, by Lemma 3.3, we have that \mathbf{A} has principal positive primitive types.

It is easy to see, that $\text{Inv}(\text{Pol}(\mathbf{A}))$ is always closed with respect to arbitrary intersections and directed unions of relations. Again using that $[\mathbf{A}]_{\text{RA}} = \overline{[\mathbf{A}]_{\text{RA}}} = \text{Inv}(\text{Pol}(\mathbf{A}))$, we conclude with Proposition 3.1, that \mathbf{A} is algebraic. \square

Corollary 4.3 (Romov [24, Thm.3.5]). *Let W be a relational algebra on a countable set. Then W is a relational clone (i.e. $W = \text{Inv}(\text{Pol}(W))$) if and only if W is closed with respect to unions of upwards directed sets of relations and with respect to arbitrary intersections of relations of equal arities.*

Proof. (\Leftarrow) Let \mathbf{C}_W be the canonical structure of W . Then $W = [\mathbf{C}_W]_{\text{RA}}$. Since W is closed with respect to unions of upwards directed sets of relations, by Proposition 3.1 we obtain that \mathbf{C}_W is algebraic. On the other hand, W is also closed with respect to arbitrary intersections, so by Lemma 3.3 it follows that \mathbf{C}_W has principal positive primitive types. Hence, by Theorem 4.2 we obtain that

$$W = [\mathbf{C}_W]_{\text{RA}} = \text{Inv}(\text{Pol}(\mathbf{C}_W)) = \text{Inv}(\text{Pol}(W)).$$

(\Rightarrow) Clear. \square

Weak oligomorphy revisited. A relational algebra W will be called *weakly oligomorphic* if $W^{(m)}$ is finite, for all $m \in \mathbb{N} \setminus \{0\}$. Clearly, a structure \mathbf{A} is weakly oligomorphic if and only if $[\mathbf{A}]_{\text{RA}}$ is weakly oligomorphic.

Proposition 4.4. *Let \mathbf{A} be a relational structure. Then the following are equivalent:*

- (1) $[\mathbf{A}]_{\text{RA}}$ is weakly oligomorphic,
- (2) $[\mathbf{A}]_{\text{RA}}$ is weakly oligomorphic,
- (3) $[\mathbf{A}]_{\text{WKA}}$ is weakly oligomorphic.

Here $[\mathbf{A}]_{\text{WKA}}$ denotes the relational algebra that consists of all relations on A that are definable by positive existential formulae over \mathbf{A} .

Proof. (1) \Rightarrow (2): For every m , we have $[\mathbf{A}]_{\text{RA}}^{(m)} \subseteq \overline{[\mathbf{A}]_{\text{RA}}^{(m)}}$. Hence, $[\mathbf{A}]_{\text{RA}}^{(m)}$ is finite, for every m .

(2) \Rightarrow (3): For every m , $[\mathbf{A}]_{\text{WKA}}^{(m)}$ is obtained from $[\mathbf{A}]_{\text{RA}}^{(m)}$ through closure with respect to finite unions. From finitely many relations, only finitely many finite unions can be formed.

(3) \Rightarrow (1): For every m , $\overline{[\mathbf{A}]_{\text{RA}}^{(m)}}$ is obtained from $[\mathbf{A}]_{\text{RA}}$ through closure with respect to arbitrary intersections. If $[\mathbf{A}]_{\text{WKA}}^{(m)}$ is finite then so is its subset $[\mathbf{A}]_{\text{RA}}^{(m)}$. However, from finitely many relations, only finitely many intersections can be formed. \square

When working over a countable basic set A , more can be said:

Proposition 4.5 (Mašulović [15]). *Let \mathbf{A} be a relational structure over a countable basic set A . Then the following are equivalent:*

- (1) \mathbf{A} is weakly oligomorphic,
- (2) $\text{Inv}(\text{End}(\mathbf{A}))$ is weakly oligomorphic,
- (3) $\text{Inv}(\text{Pol}(\mathbf{A}))$ is weakly oligomorphic.
- (4) $\text{Inv}(\text{Pol}^{(k)}(\mathbf{A}))$ is weakly oligomorphic, for some $k \geq 1$
- (5) $\text{Inv}(\text{Pol}^{(k)}(\mathbf{A}))$ is weakly oligomorphic, for all $k \geq 1$

Proof. (1) \iff (2): By Proposition 4.4, $[\mathbf{A}]_{\text{WKA}}$ is weakly oligomorphic. Now the claim follows from [20, Thm.6.15].

(2) \Rightarrow (5): For every $f \in \text{End}(\mathbf{A})$ we may define an $\hat{f} \in \text{Pol}^{(k)}(\mathbf{A})$ through $\hat{f}(x_1, \dots, x_k) := f(x_1)$. Clearly,

$$\text{Inv}(\text{End}(\mathbf{A})) = \text{Inv}(\{\hat{f} \mid f \in \text{End}(\mathbf{A})\}).$$

Hence $\text{Inv}(\text{Pol}^{(k)}(\mathbf{A})) \subseteq \text{Inv}(\text{End}(\mathbf{A}))$. So, if $\text{Inv}(\text{End}(\mathbf{A}))$ is weakly oligomorphic, then so is $\text{Inv}(\text{Pol}^{(k)}(\mathbf{A}))$. Since k was arbitrary, the claim follows.

(5) \Rightarrow (4) clear.

(4) \Rightarrow (3): Since $\text{Pol}^{(k)}(\mathbf{A}) \subseteq \text{Pol}(\mathbf{A})$, it follows that $\text{Inv}(\text{Pol}(\mathbf{A})) \subseteq \text{Inv}(\text{Pol}^{(k)}(\mathbf{A}))$. Hence, if $\text{Inv}(\text{Pol}^{(k)}(\mathbf{A}))$ is weakly oligomorphic, then so is $\text{Inv}(\text{Pol}(\mathbf{A}))$.

(3) \Rightarrow (1): For every m , the elements of $[\mathbf{A}]_{\text{RA}}^{(m)}$ are invariant under the polymorphisms of \mathbf{A} . That is, $[\mathbf{A}]_{\text{RA}} \subseteq \text{Inv}(\text{Pol}(\mathbf{A}))$. Hence, if $\text{Inv}(\text{Pol}(\mathbf{A}))$ is weakly oligomorphic, then so is $[\mathbf{A}]_{\text{RA}}$. \square

5. POLYMORPHISM-HOMOGENEITY AND THE BAKER-PIXLEY THEOREM

In this section, using the Baker-Pixley Theorem, we will prove that polymorphism-homogeneity for finite relational structures is decidable.

Near unanimity functions. Let $k \geq 3$. A k -ary function f on a set A is called *near unanimity-function* if it fulfills the following, so called, *near-unanimity-identities*:

$$\forall x, y : f(y, x, \dots, x) = f(x, y, \dots, x) = \dots = f(x, x, \dots, y) = x.$$

Proposition 5.1. *Let $\mathbf{A} = (A, (\varrho_{\mathbf{A}})_{\varrho \in L})$ be a relational structure such that the arity of the relational symbols in L is bounded above by $m \geq 2$. If every partial $(m+1)$ -ary polymorphism of \mathbf{A} can be extended to a global polymorphism, then $\text{Pol}(\mathbf{A})$ contains an $(m+1)$ -ary near unanimity-function.*

Proof. We define $f : A^{m+1} \rightarrow A$ in the following way: For all $x, y \in A$ we define

$$f(y, x, \dots, x) = f(x, y, \dots, x) = \dots = f(x, x, \dots, y) := x.$$

Our goal in the following will be to show, that the thus defined partial function is in fact a partial polymorphism of the given structure \mathbf{A} :

Take an arbitrary $\varrho \in L$ and $\bar{a}_1, \dots, \bar{a}_{\text{ar}(\varrho)} \in \text{dom } f$ (where $\bar{a}_i = (a_{i,1}, \dots, a_{i,m+1})$), such that

$$\begin{array}{cccccc} a_{1,1} & a_{1,2} & \cdots & a_{1,m+1} & \in \text{dom } f \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m+1} & \in \text{dom } f \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{\text{ar}(\varrho),1} & a_{\text{ar}(\varrho),2} & \cdots & a_{\text{ar}(\varrho),m+1} & \in \text{dom } f \\ \cap & \cap & \cdots & \cap & \\ \varrho \mathbf{A} & \varrho \mathbf{A} & \cdots & \varrho \mathbf{A} & \end{array}$$

The key is that this matrix has exactly $m + 1$ columns but at most m rows. Since every row is in the domain of f , in each row there is at most one element that occurs exactly once in this row. These elements of the matrix we shall call *distinguished*. Altogether the matrix contains at most m distinguished elements. Therefore there has to be at least one column that contains not a single distinguished entry. However, this column must be equal to

$$\begin{bmatrix} f(a_{1,1}, a_{1,2}, \dots, a_{1,m+1}) \\ f(a_{2,1}, a_{2,2}, \dots, a_{2,m+1}) \\ \vdots \\ f(a_{\text{ar}(\varrho),1}, a_{\text{ar}(\varrho),2}, \dots, a_{\text{ar}(\varrho),m+1}) \end{bmatrix}$$

whence f is a partial polymorphism. Hence, by our assumptions on \mathbf{A} , we conclude that f can be extended to a polymorphism g of \mathbf{A} . This function g is a near unanimity function. \square

Corollary 5.2. *If \mathbf{A} is a finite polymorphism-homogeneous relational structure over a finite relational signature, then $\text{Pol}(\mathbf{A})$ contains a near unanimity function.* \square

The existence of a near unanimity polymorphism in a structure \mathbf{A} has strong consequences for the clone of polymorphisms of \mathbf{A} . Let \mathbb{A} be an algebra with basic set A and with all functions from $\text{Pol}(\mathbf{A})$ as fundamental operations. Let further $V(\mathbb{A})$ be the variety generated by \mathbb{A} . If \mathbf{A} has a near unanimity polymorphism f , then there exists a term t in the language of \mathbb{A} , such that the term function defined by t in any algebra from $V(\mathbb{A})$ is a near unanimity function. Thus, the Baker-Pixley theorem can be invoked for $V(\mathbb{A})$:

Theorem 5.3 (Baker, Pixley [1]). *Let V be a variety and $d \geq 2$ be an integer. Then the following are equivalent:*

- (1) *There is a $(d+1)$ -ary term t in the language of V such that the term function of t in any algebra from V is a near unanimity function,*
- (2) *if \mathbb{A} is a subalgebra of a direct product $\mathbb{P} = \mathbb{B}_1 \times \dots \times \mathbb{B}_r$ for $r \geq d$, then \mathbb{A} is determined by all its projections onto d coordinates of this product,*
- (3) *if $\mathbb{A} \in V$ and if from r congruences $x \equiv a_i \pmod{\theta_i}$ all collections of d congruences are solvable, then all r congruences are solvable, simultaneously,*
- (4) *if $\mathbb{A} \in V$, $n \geq 1$, and if $f : A^n \multimap A$ is a partial function with a finite domain, then f extends to a term-function of \mathbb{A} if and only if every restriction of f to d or fewer elements of its domain extends to a term function of \mathbb{A} ,*
- (5) *if $\mathbb{A} \in V$, $n \geq 1$, and if $f : A^n \multimap A$ is a partial function with a finite domain, then f extends to a term function of \mathbb{A} if and only if f preserves all relations from $\text{Inv}^{(d)}(\mathbb{A})$.*

Theorem 5.4. *Let \mathbf{A} be a finite relational structure, all of whose relations have arity $\leq d$ for some $d \geq 2$. If \mathbf{A} is $|A|^d$ -polymorphism-homogeneous, then it is polymorphism-homogeneous.*

Proof. Without loss of generality, we may assume that the relational signature of \mathbf{A} is finite.

If $|A| = 1$, then the claim is trivially true. Therefore in the following we will assume that A has at least 2 elements. With this assumption we always have $|A|^d > d + 1$. By Proposition 5.1 we have that \mathbf{A} has a $(d + 1)$ -ary near unanimity polymorphism.

Let f be an n -ary partial polymorphism of \mathbf{A} such that n is larger than $|A|^d$ and such that the domain of f has m elements for some $1 \leq m \leq d$. Suppose $\text{dom}(f) = \{(a_{i,1}, \dots, a_{i,n}) \mid 1 \leq i \leq m\}$. Then the matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

has at most $|A|^m$ different columns. After removing all duplicate columns we obtain a matrix

$$\begin{bmatrix} a_{1,j_1} & a_{1,j_2} & \dots & a_{1,j_k} \\ a_{2,j_1} & a_{2,j_2} & \dots & a_{2,j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,j_1} & a_{m,j_2} & \dots & a_{m,j_k} \end{bmatrix}$$

Now, the partial function $g : A^k \dashrightarrow A$, defined by

$$g(a_{i,j_1}, a_{i,j_2}, \dots, a_{i,j_k}) := f(a_{i,1}, a_{i,2}, \dots, a_{i,n}) \quad 1 \leq i \leq m$$

is a partial polymorphism of \mathbf{A} . Since $k \leq |A|^d$, and since \mathbf{A} is $|A|^d$ -polymorphism-homogeneous, g extends to a polymorphism \hat{g} of \mathbf{A} . Now we define $\hat{f} : A^n \rightarrow A$ according to

$$\hat{f}(x_1, \dots, x_n) := \hat{g}(x_{i_1}, \dots, x_{i_k}).$$

Clearly, \hat{f} is a polymorphism of \mathbf{A} that extends f .

Hence, from the Baker-Pixley Theorem it follows that \mathbf{A} is polymorphism-homogeneous. \square

Corollary 5.5. *It is decidable whether a finite relational structure is polymorphism-homogeneous.* \square

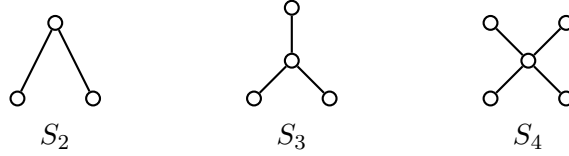
6. CLASSIFYING POLYMORPHISM-HOMOGENEOUS STRUCTURES

Until now our knowledge about polymorphism-homogeneous structures is mostly theoretical. In this section we are going to change this situation by giving a complete classifications of polymorphism-homogeneous graphs, posets, and strict posets. Finally we will recall Kaarli's classification of polymorphism-homogeneous meet-complete lattices of equivalence relations [13]. Our results partially depend on classification results of homomorphism-homogeneous structures by Cameron, Lockett, Mašulovic, and Nešetřil [2, 16, 3].

Polymorphism-homogeneous graphs. homomorphism-homogeneous structures were first searched among graphs and posets in [3]. While the characterization of homomorphism-homogeneous posets was meanwhile completed, to our knowledge such a characterization for homomorphism-homogeneous graphs is still to be given. What we know from [3] is that every countable graph that contains the Rado-graph as a spanning subgraph is homomorphism-homogeneous, and that the disconnected homomorphism-homogeneous graphs are exactly the disjoint unions of complete graphs of the same size. From [25], we know that there are countable homomorphism-homogeneous graphs that do not belong to the former classes. In this section we are going to give a complete classification of the countable polymorphism-homogeneous graphs.

When we talk about graphs, we mean simple graphs. In particular for us a graph is a pair (V, ϱ) , where V is a set of vertices and where ϱ is a symmetric, irreflexive binary relation on V . Thus, for us, homomorphisms are not allowed to contract edges to loops.

Recall that for every $k \in \mathbb{N} \setminus \{0\}$, the star graph S_k is defined to be the complete bipartite graph $K_{1,k}$.



The following Lemma is going to be the key in the classification of the connected polymorphism-homogeneous graphs.

Lemma 6.1. *Let $\mathbf{A} = (A, \varrho)$ be a graph that contains S_2 as an induced subgraph. Then for every $k > 2$ there exists an $n \in \mathbb{N} \setminus \{0\}$ such that S_k is an induced subgraph of \mathbf{A}^n .*

Proof. Let $a, b, c \in A$ such that $(a, c), (b, c) \in \varrho$, and such that $(a, b) \notin \varrho$. For $1 \leq i \leq k$ define $\bar{a}_i \in A^k$ according to:

$$\begin{aligned} \bar{a}_1 &= (b, a, a, \dots, a), \\ \bar{a}_2 &= (a, b, a, \dots, a), \\ &\vdots \\ \bar{a}_k &= (a, a, \dots, a, b). \end{aligned}$$

Clearly, the set $\{\bar{a}_1, \dots, \bar{a}_k\}$ induces an independent set in \mathbf{A}^k . Moreover, $\bar{c} := (c, c, \dots, c)$ is connected to each \bar{a}_i in \mathbf{A}^k . In other words, the set $\{\bar{a}_1, \dots, \bar{a}_k, \bar{c}\}$ induces S_k in \mathbf{A}^k . \square

Lemma 6.2. *Let \mathbf{A} be a polymorphism-homogeneous graph that contains S_2 as an induced subgraph. Then every finite subset of vertices of \mathbf{A} has a common neighbor.*

Proof. Suppose, there is a finite set $\{b_1, \dots, b_k\}$ of vertices of \mathbf{A} that has no common neighbor. By Lemma 6.1, there exists some $n \in \mathbb{N} \setminus \{0\}$ such that \mathbf{A}^n has an induced subgraph isomorphic to S_k . Let $\{\bar{a}_1, \dots, \bar{a}_k, \bar{c}\}$

be its vertex set such that \bar{c} is a common neighbor of the other vertices. For $1 \leq i \leq k$, define $\bar{b}_i \in A^n$ according to $\bar{b}_i := (b_i, \dots, b_i)$. Then the set $\{\bar{b}_1, \dots, \bar{b}_k\}$ does not have a common neighbor in \mathbf{A}^n . However, then the mapping defined by $\bar{a}_i \mapsto \bar{b}_i$ ($1 \leq i \leq k$) is a local homomorphism of \mathbf{A}^n that does not extend to an endomorphism. Hence \mathbf{A}^n is not homomorphism-homogeneous and thus \mathbf{A} is not polymorphism-homogeneous — contradiction. \square

Lemma 6.3. *Let $\mathbf{A} = (A, \varrho)$ be a graph and $k \in \mathbb{N} \setminus \{0\}$. Then $S_2 \in \text{Age}(\mathbf{A})$ if and only if $S_2 \in \text{Age}(\mathbf{A}^k)$.*

Proof. “ \Rightarrow ” Let $a, b, c \in A$ such that $(a, c), (b, c) \in \varrho$, but $(a, b) \notin \varrho$. Define $\bar{a}, \bar{b}, \bar{c} \in A^k$ according to $\bar{a} = (a, \dots, a)$, $\bar{b} = (b, \dots, b)$, $\bar{c} = (c, \dots, c)$. Then $\{\bar{a}, \bar{b}, \bar{c}\}$ induces a subgraph isomorphic to S_2 in \mathbf{A}^k .

“ \Leftarrow ” Let $\bar{a}, \bar{b}, \bar{c} \in A^k$ such that $(\bar{a}, \bar{c}) \in \varrho^k$, $(\bar{b}, \bar{c}) \in \varrho^k$, but $(\bar{a}, \bar{b}) \notin \varrho^k$. Suppose, $\bar{a} = (a_1, \dots, a_k)$, $\bar{b} = (b_1, \dots, b_k)$, and $\bar{c} = (c_1, \dots, c_k)$. Then there exists an $i \in \{1, \dots, k\}$, such that $(a_i, b_i) \notin \varrho$. However, then $\{a_i, b_i, c_i\}$ induces a subgraph isomorphic to S_2 in \mathbf{A} . \square

Proposition 6.4 (Cameron, Nešetřil [3, Prop.2.1(a).]). *A countably infinite graph \mathbf{A} contains the Rado graph as a spanning subgraph if and only if every finite set of vertices of \mathbf{A} has a common neighbor.* \square

Theorem 6.5. *A countable graph \mathbf{A} is polymorphism-homogeneous if and only if either it is a disjoint union of complete graphs of equal order or it contains the Rado graph as a spanning subgraph.*

Proof. “ \Rightarrow ” If $S_2 \notin \text{Age}(\mathbf{A})$, then \mathbf{A} is a disjoint union of complete graphs. Since \mathbf{A} is polymorphism-homogeneous, it is also homomorphism-homogeneous. Hence, it has to be the disjoint union of complete graphs of equal order.

On the other hand, if $S_2 \in \text{Age}(\mathbf{A})$, then by Lemma 6.2, and by Proposition 6.4, \mathbf{A} contains the Rado graph as a spanning subgraph.

“ \Leftarrow ” Suppose, that \mathbf{A} is the disjoint union of complete graphs of equal order n . Let $k \in \mathbb{N} \setminus \{0\}$. By Lemma 6.3, \mathbf{A}^k is the disjoint union of complete graphs. Let $\{\bar{a}_1, \dots, \bar{a}_m\}$ be the vertex set of a connected component of \mathbf{A}^k . Further suppose that $\bar{a}_i = (a_{i,1}, \dots, a_{i,k})$, for $1 \leq i \leq m$. Then all the elements of the set $\{a_{1,j}, \dots, a_{m,j}\}$ are mutually distinct. Hence $m \leq n$. If $m < n$, then for every $j \in \{1, \dots, k\}$, there exists an element b_j from the connected component of \mathbf{A} in which lies $a_{1,j}$, that is not in $\{a_{1,j}, \dots, a_{m,j}\}$. However, then with $\bar{b} = (b_1, \dots, b_k)$, we get that $\{\bar{a}_1, \dots, \bar{a}_m, \bar{b}\}$ forms a complete subgraph of \mathbf{A}^k — contradiction. Hence, all connected components of \mathbf{A}^k are complete graphs of order n , whence we have that \mathbf{A}^k is homomorphism-homogeneous. We conclude that \mathbf{A} is polymorphism-homogeneous.

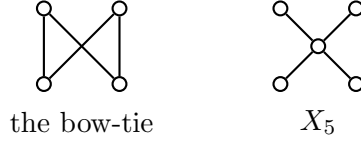
Suppose now that \mathbf{A} has the Rado graph as a spanning subgraph. By Proposition 6.4, every finite set of vertices of \mathbf{A} has a common neighbor. Let $k \in \mathbb{N} \setminus \{0\}$, and let $\{\bar{a}_1, \dots, \bar{a}_n\}$ be a set of vertices of \mathbf{A}^k . For $1 \leq i \leq n$, suppose, $\bar{a}_i = (a_{i,1}, \dots, a_{i,k})$. For $j \in \{1, \dots, k\}$, let c_j be a common neighbor of $\{a_{1,j}, \dots, a_{n,j}\}$, and set $\bar{c} := (c_1, \dots, c_k)$. Then \bar{c} is a common neighbor of $\{\bar{a}_1, \dots, \bar{a}_n\}$. Hence, by Proposition 6.4, \mathbf{A}^k has the Rado graph

as a spanning subgraph. In particular, \mathbf{A}^k is homomorphism-homogeneous. It follows that \mathbf{A} is polymorphism-homogeneous. \square

Polymorphism-homogeneous posets. A poset is a relational structure $\mathbf{A} = (A, \leq)$ where \leq is a binary reflexive, antisymmetric transitive relation. \mathbf{A} is called *trivial* if $a \leq b$ implies $a = b$ — in other words, it is an anti-chain. For $X, Y \subseteq A$ and $x, y \in A$ we write

$$\begin{aligned} x \leq Y & \quad \text{if} \quad \forall y \in Y (x \leq y), \\ X \leq y & \quad \text{if} \quad \forall x \in X (x \leq y), \\ X \leq Y & \quad \text{if} \quad \forall x \in X \forall y \in Y (x \leq y). \end{aligned}$$

We distinguish the following posets:



Definition 6.6. Let $\mathbf{A} = (A, \leq)$ be a poset. Then \mathbf{A} is called *locally bounded* if for every finite subset B of A there exist $c, d \in A$ such that $c \leq B \leq d$.

Definition 6.7. Let $\mathbf{A} = (A, \leq)$ be a poset. Then \mathbf{A} is called *X_5 -dense* if for all mutually distinct $a_1, a_2, a_3, a_4 \in A$ with $\{a_1, a_2\} \leq \{a_3, a_4\}$, there exists a $c \in A$ such that $\{a_1, a_2\} \leq c \leq \{a_3, a_4\}$.

The homomorphism-homogeneous posets were completely characterized by Mašulović [16] and, independently, by Cameron and Lockett [2]:

Theorem 6.8 (Mašulović [16, Thm.4.5]). *A poset \mathbf{A} is homomorphism-homogeneous if and only if*

- (1) *every connected component of \mathbf{A} is a chain, or*
- (2) *\mathbf{A} is a tree, or*
- (3) *\mathbf{A} is a dual tree, or*
- (4) *\mathbf{A} is locally bounded and $X_5 \notin \text{Age}(\mathbf{A})$, or*
- (5) *\mathbf{A} is locally bounded and X_5 -dense.*

Lemma 6.9. *Let $\mathbf{A} = (A, \leq)$ be a poset and let $k \in \mathbb{N} \setminus \{0\}$. Then \mathbf{A} is X_5 -dense if and only if \mathbf{A}^k is X_5 -dense.*

Proof. “ \Rightarrow ” Let $k \geq 1$, and let $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4 \in A^k$ such that $\{\bar{a}_1, \bar{a}_2\} \leq \{\bar{a}_3, \bar{a}_4\}$ in \mathbf{A}^k . Let us say $\bar{a}_i = (a_{i,1}, \dots, a_{i,k})$ for $i \in \{1, 2, 3, 4\}$. Then for all $j \in \{1, \dots, k\}$ we have $\{a_{1,j}, a_{2,j}\} \leq \{a_{3,j}, a_{4,j}\}$ in \mathbf{A} . By the assumption, for every $j \in \{1, \dots, k\}$ there exists a c_j such that $\{a_{1,j}, a_{2,j}\} \leq c_j \leq \{a_{3,j}, a_{4,j}\}$. But then, with $\bar{c} = (c_1, \dots, c_k)$, we also have $\{\bar{a}_1, \bar{a}_2\} \leq \bar{c} \leq \{\bar{a}_3, \bar{a}_4\}$. Hence \mathbf{A}^k is X_5 -dense.

“ \Leftarrow ” Let $a_1, a_2, a_3, a_4 \in A$ such that $\{a_1, a_2\} \leq \{a_3, a_4\}$ in \mathbf{A} . Define $\bar{a}_i \in A^k$ according to $\bar{a}_i = (a_i, \dots, a_i)$, where $i \in \{1, 2, 3, 4\}$. Clearly, we have $\{\bar{a}_1, \bar{a}_2\} \leq \{\bar{a}_3, \bar{a}_4\}$ in \mathbf{A}^k . Thus, by the assumption, there exists $\bar{c} \in A^k$ such that $\{\bar{a}_1, \bar{a}_2\} \leq \bar{c} \leq \{\bar{a}_3, \bar{a}_4\}$ in \mathbf{A}^k . Suppose $\bar{c} = (c_1, \dots, c_k)$. Then $\{a_1, a_2\} \leq c_1 \leq \{a_3, a_4\}$ in \mathbf{A} . Hence \mathbf{A} is X_5 -dense. \square

Lemma 6.10. *Let $\mathbf{A} = (A, \leq)$ be a poset and let $k \in \mathbb{N} \setminus \{0\}$. Then \mathbf{A} is locally bounded if and only if \mathbf{A}^k is locally bounded.* \square

Proof. “ \Rightarrow ” Let $\{\bar{a}_1, \dots, \bar{a}_m\} \subseteq A^k$, where $\bar{a}_i = (a_{i,1}, \dots, a_{i,k})$ for $i \in \{1, \dots, m\}$. By the assumption, we have that for every $j \in \{1, \dots, k\}$ there exist $c_j, d_j \in A$ such that $c_j \leq \{a_{1,j}, \dots, a_{m,j}\} \leq d_j$. Define $\bar{c} := (c_1, \dots, c_k)$ and $\bar{d} := (d_1, \dots, d_k)$. Then $\bar{c} \leq \{\bar{a}_1, \dots, \bar{a}_m\} \leq \bar{d}$. Hence, \mathbf{A}^k is locally bounded.

“ \Leftarrow ” Let $\{a_1, \dots, a_m\} \subseteq A$. For $i \in \{1, \dots, m\}$ define $\bar{a}_i \in A^k$ according to $\bar{a}_i := (a_i, \dots, a_i)$. By the assumption, there exist $\bar{c}, \bar{d} \in A^k$ such that $\bar{c} \leq \{\bar{a}_1, \dots, \bar{a}_m\} \leq \bar{d}$. Suppose, $\bar{c} = (c_1, \dots, c_k)$, $\bar{d} = (d_1, \dots, d_k)$. Then we have $c_1 \leq \{a_1, \dots, a_m\} \leq d_1$. Hence, \mathbf{A} is locally bounded. \square

Lemma 6.11. *Let $\mathbf{A} = (A, \leq)$ be a poset that contains a chain of length l . Then for every l -element poset \mathbf{B} there exists a $k \in \mathbb{N} \setminus \{0\}$ such that $\mathbf{B} \in \text{Age}(\mathbf{A}^k)$.*

Proof. By the Dushnik-Miller Theorem [8], the order relation of \mathbf{B} is equal to the intersection of a collection of linear order relations on B . The smallest possible size of such a collection is called the order dimension of \mathbf{B} . In other words, if the order dimension of \mathbf{B} is equal to k , then there exists an $l \times k$ -matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{l,1} & a_{l,2} & \dots & a_{l,k} \end{pmatrix}$$

with entries from B such that

- (1) $\forall i \in \{1, \dots, k\} : \{a_{1,i}, \dots, a_{l,i}\} = B$,
- (2) $\forall a, b \in B : a \leq b \iff \left(\forall j \in \{1, \dots, k\} : h_j(a) \leq h_j(b) \right)$,

where $h_j(x) = i$ whenever $a_{i,j} = x$.

Let $c_1 < c_2 < \dots < c_l$ be a chain of length l in \mathbf{A} . Define $f : B \rightarrow A^m$ by $a \mapsto (c_{h_1(a)}, \dots, c_{h_k(a)})$. Then, by the second property of the matrix M , we have that f is actually an embedding of \mathbf{B} into \mathbf{A}^k . In particular, $\mathbf{B} \in \text{Age}(\mathbf{A}^k)$. \square

Lemma 6.12. *Let $\mathbf{A} = (A, \leq)$ be a poset that is not an antichain. Then for every finite poset \mathbf{B} there exists a $k \in \mathbb{N} \setminus \{0\}$ such that $\mathbf{B} \in \text{Age}(\mathbf{A}^k)$.* \square

Proof. Since \mathbf{A} is not an antichain, it contains at least two elements. Suppose that \mathbf{B} has l elements (we may assume that $l > 1$ since otherwise nothing needs to be proved). Let $a, b \in A$ such that $a < b$. Then \mathbf{A}^{l-1} contains a chain of length l — namely

$$(a, a, \dots, a) < (a, a, \dots, a, b) < (a, a, \dots, b, b) < \dots < (b, b, \dots, b).$$

Hence, by Lemma 6.11, there exists some m such that $\mathbf{B} \in \text{Age}((\mathbf{A}^{l-1})^m)$. Since $(\mathbf{A}^{l-1})^m \cong \mathbf{A}^{(l-1) \cdot m}$, with $k := (l-1)m$, we have $\mathbf{B} \in \text{Age}(\mathbf{A}^k)$. \square

Lemma 6.13. *Let $\mathbf{A} = (A, \leq)$ be a polymorphism-homogeneous poset that is not an antichain. Then \mathbf{A} is locally bounded.*

Proof. Suppose on the contrary that \mathbf{A} is not locally bounded. Without loss of generality assume that \mathbf{A} is not upwards directed. Let a, b be two elements of \mathbf{A} without a joint upper bound. Since \mathbf{A} is not an antichain, by Lemma 6.12, there exists a k , such that in \mathbf{A}^k there exist two non-comparable elements \bar{c}, \bar{d} that have a joint upper bound \bar{u} . Let $\bar{a}, \bar{b} \in A^k$ defined according to $\bar{a} := (a, \dots, a)$ and $\bar{b} := (b, \dots, b)$. Then \bar{a} and \bar{b} have no joint upper bound in \mathbf{A}^k . It follows that \mathbf{A}^k is not homomorphism-homogeneous. Hence \mathbf{A} is not polymorphism-homogeneous — contradiction. \square

Lemma 6.14. *Let $\mathbf{A} = (A, \leq)$ be a polymorphism-homogeneous poset. Then \mathbf{A} is X_5 -dense.*

Proof. Suppose on the contrary that \mathbf{A} is not X_5 -dense. Then there exist four elements $a_1, a_2, a_3, a_4 \in A$ such that $\{a_1, a_2\} \leq \{a_3, a_4\}$, but there is no $c \in A$ such that $\{a_1, a_2\} \leq c \leq \{a_3, a_4\}$. In this case, $\{a_1, a_2, a_3, a_4\}$ induce a substructure of \mathbf{A} that is isomorphic to the bow-tie. In particular, \mathbf{A} is not an antichain. From Lemma 6.12 it follows that there is a $k \in \mathbb{N} \setminus \{0\}$ such that there are $\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4, \bar{d} \in A^k$ which fulfill $\{\bar{b}_1, \bar{b}_2\} \leq \bar{d} \leq \{\bar{b}_3, \bar{b}_4\}$. Now, for $i \in \{1, 2, 3, 4\}$ define $\bar{a}_i \in A^k$ according to $\bar{a}_i = (a_i, \dots, a_i)$. Then there is no $\bar{c} \in A^k$ such that $\{\bar{a}_1, \bar{a}_2\} \leq \bar{c} \leq \{\bar{a}_3, \bar{a}_4\}$. Note that the mapping $f : \bar{b}_i \mapsto \bar{a}_i$ ($i \in \{1, 2, 3, 4\}$) is a local homomorphism of \mathbf{A}^k which can not be extended to an endomorphism of \mathbf{A}^k . We conclude that \mathbf{A}^k is not homomorphism-homogeneous and thus \mathbf{A} is not polymorphism-homogeneous — contradiction. \square

Now we are ready to give a complete characterization of the polymorphism-homogeneous posets. Note that the characterization is independent of the cardinality of the posets in question.

Theorem 6.15. *A partially ordered set $\mathbf{A} = (A, \leq)$ is polymorphism-homogeneous if and only if either it is an antichain, or it is locally bounded and X_5 -dense.*

We obtain the following characterization for finite polymorphism-homogeneous posets:

Corollary 6.16. *A finite poset \mathbf{A} is polymorphism-homogeneous if and only if it is an antichain or a lattice-order.* \square

Polymorphism-homogeneous strict posets. A strict poset is a relational structure $\mathbf{A} = (A, <)$, where $<$ is a binary irreflexive asymmetric, and transitive relation. For a set $M \subseteq A$, by $\mathbf{A}_{<M}$ and $\mathbf{A}_{>M}$ we denote the set of lower bounds and the set of upper bounds of M in \mathbf{A} , respectively. We say that \mathbf{A} is locally bounded if every finite subset of A has a lower and an upper bound. Moreover, \mathbf{A} is called X_5 -dense if for every $\{a_1, a_2, a_3, a_4\} \in A$ with $\{a_1, a_2\} < \{a_3, a_4\}$ there exists a $c \in A$ such that $\{a_1, a_2\} < c < \{a_3, a_4\}$.

The countable homomorphism-homogeneous strict posets were completely characterized by Cameron and Lockett:

Theorem 6.17 (Cameron, Lockett [2, Prop.15]). *A countable strict poset $\mathbf{A} = (A, <)$ is homomorphism-homogeneous if and only if*

- (1) *it is an antichain, or*

- (2) it is a disjoint union of copies of $(\mathbb{Q}, <)$, or
- (3) it is a tree with no minimal elements such that for all finite $Q \subseteq A$, $\mathbf{A}_{<Q}$ has no maximal elements, or
- (4) it is a dual tree with no maximal elements such that for all finite $Q \subseteq A$, $\mathbf{A}_{>Q}$ has no minimal elements, or
- (5) it is locally bounded, for all finite $X \subseteq A$, $\mathbf{A}_{<X}$ has no maximal elements and $\mathbf{A}_{>X}$ has no minimal elements, and it is X_5 -dense, or
- (6) it is locally bounded, for all finite $X \subseteq A$, $\mathbf{A}_{<X}$ has no maximal elements and $\mathbf{A}_{>X}$ has no minimal elements, and $X_5 \notin \text{Age}(\mathbf{A})$.

It was shown in [2, Prop.13] that the posets from Theorem 6.17(5) are exactly the extensions of the countable universal homogeneous poset $\mathbb{U} = (U, <)$.

Lemma 6.18. *Let $\mathbf{A} = (A, <)$ be a strict poset and let $k \in \mathbb{N} \setminus \{0\}$. Then \mathbf{A} is X_5 -dense if and only if \mathbf{A}^k is X_5 -dense.*

Proof. Cf. the proof of Lemma 6.9. □

Lemma 6.19. *Let $\mathbf{A} = (A, <)$ be a strict poset and let $k \in \mathbb{N} \setminus \{0\}$. Then \mathbf{A} is locally bounded if and only if \mathbf{A}^k is locally bounded.* □

Proof. Cf. the proof of Lemma 6.10. □

Lemma 6.20. *Let $\mathbf{A} = (A, <)$ be a strict poset that contains a chain of length l . Then for every l -element poset \mathbf{B} there exists a $k \in \mathbb{N} \setminus \{0\}$ such that $\mathbf{B} \in \text{Age}(\mathbf{A}^k)$.*

Proof. Cf. the proof of Lemma 6.11. □

Lemma 6.21. *Let \mathbf{A} be a polymorphism-homogeneous strict poset that is not an antichain. Then \mathbf{A} is locally bounded and X_5 -dense.*

Proof. Essentially the same proofs as in Lemma 6.13 and Lemma 6.14 work. Instead of using Lemma 6.12, we use that every homomorphism-homogeneous strict poset that is not an antichain contains a countably infinite chain, in conjunction with Lemma 6.11. □

Now we are ready to give a complete characterization of the countable polymorphism-homogeneous strict posets:

Theorem 6.22. *A countable strict partially ordered set $\mathbf{A} = (A, <)$ is polymorphism-homogeneous if and only if either it is an antichain, or it is an extension of the countable universal homogeneous poset $\mathbb{U} = (U, <)$.*

Proof. “ \Leftarrow ” Clearly, if \mathbf{A} is an antichain, then it is polymorphism-homogeneous.

By Lemma 6.21, it remains only to show that every extension of \mathbb{U} is polymorphism-homogeneous. So suppose that \mathbf{A} is an extension of \mathbb{U} . We will show that for every $k \in \mathbb{N} \setminus \{0\}$, \mathbf{A}^k is an extension of \mathbb{U} .

Let $X \subseteq A^k$ be finite. Assume $X = (\bar{x}_1, \dots, \bar{x}_n)$ and for $1 \leq i \leq n$ suppose $\bar{x} = (x_{i,1}, \dots, x_{i,k})$. Let $\bar{y} = (y_1, \dots, y_k)$ be a lower bound of X . Then for every $1 \leq j \leq k$, y_j is a lower bound of $\{x_{1,j}, \dots, x_{n,j}\}$. But by the assumption, that \mathbf{A} is an extension of \mathbb{U} , there exist for every $j \in$

$\{1, \dots, k\}$ some $\hat{y}_j > y_j$ that is a lower bound of $\{x_{1,j}, \dots, x_{n,j}\}$. However, then $(\hat{y}_1, \dots, \hat{y}_k)$ is a lower bound of X that is strictly above \bar{y} .

Analogously it is proved that X has no minimal upper bound.

Now, from Lemmas 6.18, and 6.19 it follows that all finite powers of \mathbf{A} are homomorphism-homogeneous. Thus, \mathbf{A} is polymorphism-homogeneous.

“ \Leftarrow ” This is Lemma 6.21. \square

Polymorphism-homogeneous lattices of equivalence relations. Let A be a set and let $\mathcal{E}(A)$ be the set of all equivalence relations on A . Then $\mathcal{E}(A)$, ordered by inclusion forms a complete lattice, in which the infimum of a set of equivalence relations is their intersection, and the supremum is the transitive closure of their union.

A sublattice \mathcal{L} of $\mathcal{E}(A)$ is called *meet-complete* if it is closed with respect to arbitrary intersections. It is called *arithmetical* if

- (1) $\forall \theta_1, \theta_2 \in \mathcal{L} : \theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$,
- (2) $\forall \theta_1, \theta_2, \theta_3 \in \mathcal{L} : \theta_1 \wedge (\theta_2 \vee \theta_3) = (\theta_1 \wedge \theta_2) \vee (\theta_1 \wedge \theta_3)$

We will call a sublattice \mathcal{L} of $\mathcal{E}(A)$ polymorphism-homogeneous if its canonical structure $\mathbf{C}_{\mathcal{L}}$ is polymorphism-homogeneous. The following beautiful characterization of polymorphism-homogeneous meet-complete sublattices of $\mathcal{E}(A)$ is due to Kaarli [13]:

Theorem 6.23 (Kaarli [13, Thm.3]). *Let A be a countable set and let \mathcal{L} be a meet-complete sublattice of $\mathcal{E}(A)$. Then \mathcal{L} is polymorphism-homogeneous if and only if it is arithmetical.*

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